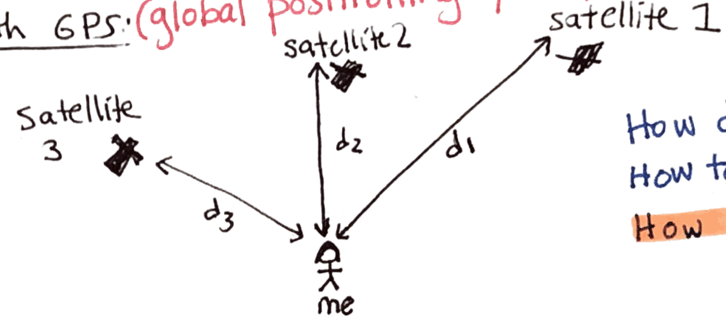
Module 3 Lec 1

- positioning / locationing
- norms
- inner products

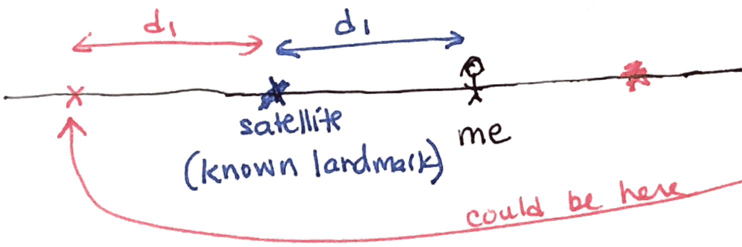
Positioning with GPS: (global positioning system)



Where am I?

How do dist. help positioning?
How to meas. distances?
How many satellites do I need?

Let's simplify to $\boxed{1D}$:

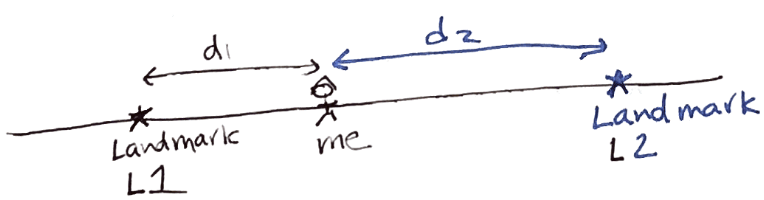


Assume I can meas. d_1, does this uniquely give position?

No, cause could be ~~dn~~ to direction on either side...

What else do I need? direction (vector!)

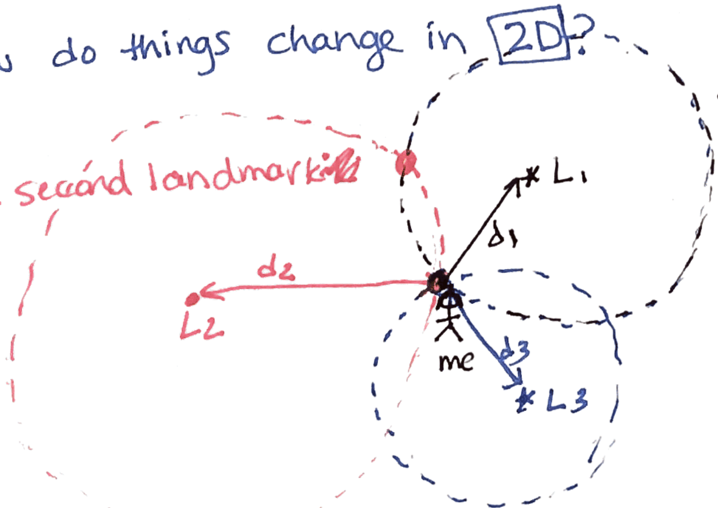
OR? A second known landmark!



What are requirements on ~~the~~ Landmark L2?

- known
- $d_1 \neq d_2$

How do things change in $\boxed{2D}$?



← I could be anywhere on this circle!
→ More ambiguity in 2D.

Add a second landmark!

Now 2 possible solutions!
(intersects of circles)

Now what to do?
add a 3rd Landmark!

Now correct position is only solution!

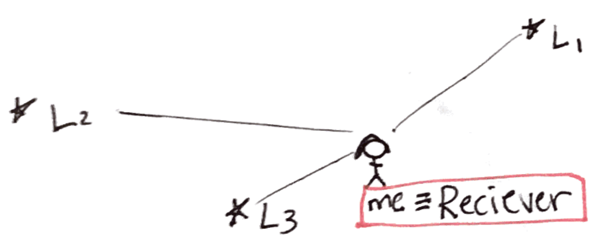
Could I do it in only two? Not in general

Need: - 3 known landmarks → "~~triangulation~~" "trilateration"
- dist. from each
- NOT collinear (prove later)

Problem: How can I get the distance measurement?

- ruler? not practical
- send a robot at speed v, see how long it takes ($d=vt$)?
  robots are not practical
- send a signal wave:
  - EM waves (light, microwave) $\boxed{\text{GPS:}}$ 3×10^8 m/s
  - Sound waves ~~340 m/s~~ $\boxed{\text{APS:}}$ 340 m/s
    ↳ we'll do this one in lab
    ↑ acoustic pos. system

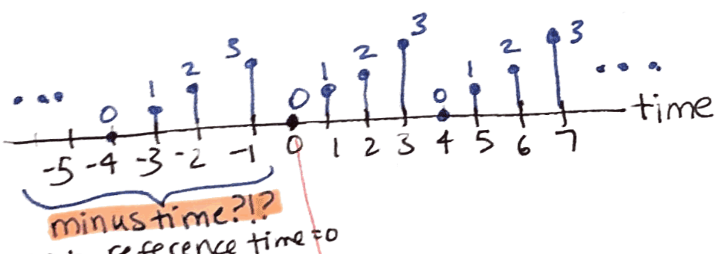
* Assume for now we have clocks @ Landmarks and reciever (me) that are synced.



Landmarks ≡ beacons

GPS/APS: the beacons "sing songs" in a periodic loop, transmitting in all directions equally
↳ send signals
one-way
& the reciever "listens to" the songs (from many beacons may be)

Example:

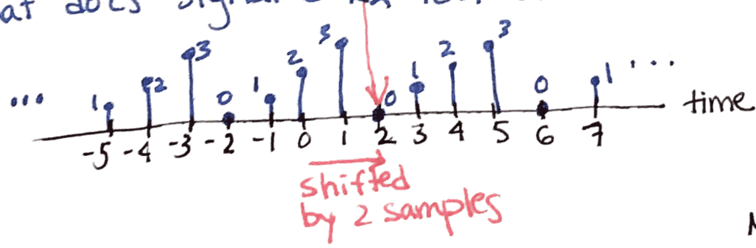


"Song" is 0, 1, 2, 3, 0, 1, 2, 3, 0....

time = samples
↳ assume discrete time.

So, if signal takes 2 samples of time to get from beacon to reciever (R_x) then it will be delayed by 2 samples.

What does signal @ R_x look like?



What if delay = 4 samples?
Cannot distinguish from 0, 8, 12, 16...
To avoid this, use ENGINEERING!

How?
Make sure max distance is song length.
i.e. if 4 samples = 4 seconds and $v = 100$ m/s, then max. dist. w̄ no ambiguity is $d = vt$
$= (100\text{ m/s})(4\text{s})$
$= 400$ m

GPS: 1ms → 300km
APS: 230ms → 55m

Summary of Knowns:

1) Time
2) Song
3) When song started
4) Beacon positions

Write beacon song as a vector:

$$\vec{s} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} s(0) \\ s(1) \\ s(2) \\ s(3) \end{bmatrix}$$

don't need to include repeats/copies since periodic. (no new info)

Song is N-periodic (N samples is period of song). Here $N=4$

Recieved signal is delayed:

$$\vec{r} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r(0) \\ r(1) \\ r(2) \\ r(3) \end{bmatrix}$$

define shifted vectors:

$$\vec{s}_0 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{s}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \boxed{\vec{s}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}} \quad \vec{s}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

index of shift

this is what we recieve after 2 sample delay.

How many possible shifts? $\underbrace{\{0, 1, 2 \ldots N-1\}}_{N \text{ total}}$ system is designed so shift $\geq N$ not allowed

How should I find the shift, $k=2$, from the recieved signal?

delay # samples

- just look at $r(0)$ and see which entry of $\vec{s}$ it is? not very robust
- compare against dictionary of shifted vectors $\{\vec{s}_0, \vec{s}_1, \vec{s}_2, \vec{s}_3\}$? how to 'compare' may not be equal if errors/noise
- what we will do is

$$\left.\begin{array}{l} \|\vec{r} - \vec{s}_0\| \\ \|\vec{r} - \vec{s}_1\| \\ \|\vec{r} - \vec{s}_2\| \\ \|\vec{r} - \vec{s}_3\| \end{array}\right\}$$

figure out which one is smallest value $\rightarrow$ delay k

$$\|\vec{r} - \vec{s}_k\| = 0$$

What is this notation?

"norm"

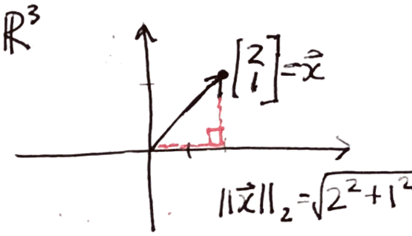
(Euclidean) norm of a vector $\vec{x}$ is $\boxed{\|\vec{x}\|_2 = \|\vec{x}\|} = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$

→ if no subscript → 2 norm

*general: p-norm is $\|\vec{x}\|_p = (x_1^p + x_2^p + \ldots x_n^p)^{1/p}$

Intuitively, what is norm?

→ magnitude (length) of vector → distance in $\mathbb{R}^2, \mathbb{R}^3$



$\|\vec{x}\|_2 = \sqrt{2^2 + 1^2} = \sqrt{5}$

So we're looking for min. dist between $\vec{r}$ and $\vec{s}_i$ → ERROR

~~$\|\vec{e}\| = \|\vec{r} - \vec{s}_i\|^2$~~

Norm has square roots, which are annoying, so just use error^2

Why can I do that? doesn't change which is min

$$\underbrace{\|\vec e_i\|^2}_{\text{error}} = \|\vec r - \vec s_i\|^2 = (\vec r - \vec s_i)\cdot(\vec r - \vec s_i)$$

$$= (\vec r - \vec s_i)^T(\vec r - \vec s_i)$$

$$= \vec r^{\,T}\vec r - (\vec s_i^{\,T}\vec r) - (\vec r^{\,T}\vec s_i) + \vec s_i^{\,T}\vec s_i$$

$$= \underbrace{\|\vec r\|^2}_{\text{NO}} + \underbrace{\|\vec s_i\|^2}_{\text{does it?}} - \underbrace{(\vec s_i^{\,T}\vec r + \vec r^{\,T}\vec s_i)}_{\text{yes}}$$

Which terms change w/ j?

No, cause shifted version of same thing → same length

Check: $\|\vec s\|^2 = s(0)^2 + s(1)^2 + \dots s(n-1)^2$

ex. $\|\vec s_1\|^2 = s(-1)^2 + s(0)^2 + \dots s(n-2)^2 = \|\vec s\|^2$

Want to minimize: $\|\vec e_i\|^2 = 2\|s\|^2 - \underbrace{(\vec s_i^{\,T}\vec r - \vec r^{\,T}\vec s_i)}_{\text{Equiv. to MAXimize this}}$

Want to maximize $\vec s_i^{\,T}\vec r + \vec r^{\,T}\vec s_i$

What is this?

if $\vec a = \vec s_i = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec b = \vec r = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, then $\vec a^T\vec b = (a_1b_1 + a_2b_2 + \dots a_nb_n)$

element-wise multiplication

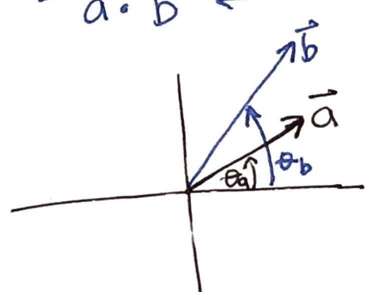
Euclidian $\boxed{\text{"Inner Product"}}$ $\vec a^T\vec b \equiv \langle \vec a, \vec b\rangle$ aka "dot product"

$= \vec a \cdot \vec b$

In $\boxed{2D}$:

$\vec a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ length of $\vec a = \sqrt{\langle \vec a, \vec a\rangle}$ = norm! $\|\vec a\|^2 = a_1^2 + a_2^2$ θ_a angle

$\vec b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ $\|\vec b\|^2 = b_1^2 + b_2^2$ angle θ_b



Can also write: $\vec a = \begin{bmatrix} \|\vec a\|\cos\theta_a \\ \|\vec a\|\sin\theta_a \end{bmatrix}$ $\vec b = \begin{bmatrix} \|\vec b\|\cos\theta_b \\ \|\vec b\|\sin\theta_b \end{bmatrix}$

Let's compute inner product:

$$\langle \vec a, \vec b\rangle = a_1b_1 + a_2b_2$$

$$= \|\vec a\|\cos\theta_a\,\|\vec b\|\cos\theta_b + \|\vec a\|\sin\theta_a\,\|\vec b\|\sin\theta_b$$

trig. exp.

$$= \|\vec a\|\,\|\vec b\|\cos(\theta_a - \theta_b)$$

difference in angles

so measures how "aligned/similar" vectors are

when collinear → large

when $\perp$ → zero

Some rules about inner products:

① $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

② $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

③ $\langle \alpha \vec{x}, \vec{y} \rangle = \alpha \langle \vec{x}, \vec{y} \rangle$

④ $\langle \vec{x}, \vec{x} \rangle \geq 0$

⑤ $\langle \vec{x}, \vec{x} \rangle = 0$ <u>iff</u> $\vec{x} = 0$

What does $\langle \vec{a}, \vec{b} \rangle = -1$ mean? anti-similar (opp. dir.)

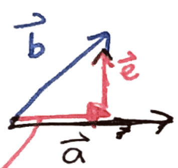
What is max. of inner product?

$\langle \vec{a}, \vec{b} \rangle \leq \|\vec{a}\| \|\vec{b}\|$ $\boxed{\text{Cauchy-Schwarz Inequality}}$

$\boxed{\text{Projections}}$

"project $\vec{b}$ onto subspace spanned by $\vec{a}$"

Scary! → go to geometry



$\hat{b}$ is projection

projected vector is component ~~alon~~ of $\vec{b}$ along direction of $\vec{a}$

What does it have to do w inner product?

Looking for <u>collinear</u> component → largest inner prod.

<u>OR</u> look for $\perp$ error → inner prod. = 0

$\vec{e} \perp b$

Scaled version of $\vec{a}$: $\hat{b} = \alpha \vec{a}$

Perp. vectors have inner prod. = 0:

$\langle \vec{e}, \vec{a} \rangle = 0$

$\langle \vec{b} - \hat{b}, \vec{a} \rangle = 0$

$\langle \vec{b}, \vec{a} \rangle - \langle \hat{b}, \vec{a} \rangle = 0$

$\langle \vec{b}, \vec{a} \rangle = \langle \alpha \vec{a}, \vec{a} \rangle$

$\langle \vec{b}, \vec{a} \rangle = \alpha \langle \vec{a}, \vec{a} \rangle \to \|\vec{a}\|^2$

$\alpha = \dfrac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}$

So then

$\hat{b} = \boxed{\dfrac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}} \vec{a}$

Scaling factor